Methodology Note: DD CEV LMM

Actuarial Function

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# Executive Summary

Adverse interest rate movements are a major risk concern to Aviva France’s portfolio. It is of particular importance to properly model and manage these risks. Model selection and calibration are therefore key to any interest rate risk management endeavour.

The anchor model for this paper is an extension of the *Libor Market Model (LMM).* This is because the LMM can be calibrated on discretely discounted rates directly observable in the market.

The standard LMM, however, does not properly capture the volatility dynamics of complex financial instruments. It is for this reason that an extension of the model is used.

We expound on both the LMM and its extension, the Displaced Diffusion Constant Elasticity of Variance LMM (DD CEV LMM) and establish pricing and calibration parameters.

This will be done in the following key sections:

1. **Projection:** We define the forward diffusion under the spot Libor Measure.

* We also proceed to define an interator allowing us to project zero coupon bonds.
* Finally, the section expounds on modelling the deflator.

1. **Swaption Pricing:** We provide the forward swap diffusion and the Chi-square closed form pricer.
2. **Calibration:**  We expound on the Levenberg Marquardt algorithm that will serve as the optimization function.

# Projection

The Displaced Diffusion CEV LFM (DD CEV LFM) is a stochastic model that models the forward/ forward swap rate as a sum of 2 non correlated DD CEV models.

The key particularity of this model is the conditional volatility structure. We know anecdotally that implied volatility tends to move in the opposite direction to rates. This model allows us to replicate this relationship.

The DD CEV LFM also allows for the modelling of the forward swap rate under a forward rate framework using an industry standard approximation called the freezing technique and pricing of swaptions using a closed form formula. This allows us to price caps, floors and swaptions under a single model.

## Forward Diffusion under Spot LIBOR Measure

Two of the most common market models are the **Lognormal Forward Model(LFM)** and the **Lognormal Forward-Swap model.** The LFM has as its primary assumption that the forward rates are lognormally distributed. The LSM has as its primary assumption that swap rates are lognormally distributed.

We define the forward rate at between as for a set of dates as:

where is the price of a zero coupon bond maturing at and is the time difference between and .

We define the diffusion of the forward rate under the spot Libor measure to be:

where is a 2-dimensional Brownian motion.

We also note that the component Brownian motions are independent i.e.

## Projection of Zero Coupons

We recall the definition of the forward rate and use a recurrence relation to find the value of our zero coupons.

The forward rate is defined as:

Therefore:

We can use this recursive definition to find the zero coupon price at :

## Projection of the Discount Factor

Seeing that the LMM is not a short rate model, particular attention should be given to the discount factor .

In the case of the LMM, the discount factor is modelled as the product of zero coupon bonds of the shortest possible maturity.

This implies defining the discount factor at time 0 to 1 i.e. .

We then recursively define:

# Swaption Pricing

In this section, we obtain the forward swap rate diffusion under the LFM framework. This methodology is derived from the standard LFM.

We use the already documented CEV LFM methodology to obtain a similar framework for the DD CEV LFM.

## Forward Swap Diffusion

In this section, we use the LFM model (defined in [Forward Diffusion under Spot LIBOR Measure](#Xc2702e0f0df91de129717d726cdf294b3e486c4)) to obtain the forward swap rate diffusion.

We divide this section into the following parts:

1. **Swap Dynamics:** We decompose swap rates as a combination of forward rates;
2. **Forward Swap Diffusion:** We define, under the decomposition in (1), a diffusion for the forward swap rate;
3. **Freezing Technique:** We apply the freezing technique to obtain an approximation of the diffusion.

### Swap Dynamics

We begin with the valuation of an interest rate swap with a fixed rate , notional maturing at with representing the time difference between and . At initiation, the value of the floating leg is the par value and the value of the swap is 0. This implies that:

We can show that the value of is:

Under this definition, we can demonstrate that the forward swap initializing at and maturing at is:

We observe, however, that we can expand the numerator such that:

We multiply the numerator and denominator by and recall the definition of the forward to obtain:

Which we can simplify to:

## Forward Swap Diffusion

### Preliminary Note

Before going further, we note that this section is an application of the *Extended Market Model* framework by Andersen & Andreasen (1998) for volatility functions defined as:

The properties of the function are also elaborated in Section 3 of the paper.

### From forward swap rate to forward swap diffusion

From the definition of the forward swap above and an application of Ito’s lemma, we can use the DD CEV LFM framework to define a diffusion for the forward swap:

where is Kronecker’s Delta and[[1]](#footnote-31):

We can simplify our notation in the following manner:

### Freezing Technique

From the last section, we defined the diffusion of the swap under the DD CEV LFM framework to be:

This allows us to diffuse the forward swap rate under any measure. Selecting **the spot LIBOR measure**, our diffusion becomes:

Under the forward swap measure, the diffusion of the forward swap rate is a martingale. Using this fact, we can define the diffusion of the swap rate under , the forward swap measure:

Since, the diffusion is driftless, we obtain:

Similarly, we can multiply both the numerator and denominator by to obtain:

The freezing technique entails setting to . These weights are therefore frozen in time. Our final diffusion is, therefore an approximation of the diffusion under the CEV LSM framework allowing us to price both caps and swaptions.

The definitive approximation of the forward swap rate diffusion, therefore is:

## Swaptions Pricing (Chi-Square)

We have obtained a diffusion for the forward swap rate in the above section. In this section, we use this diffusion to arrive at an analytical pricing formula for the DD CEV LFM.

We divide this section into a number of parts:

1. **DD CEV LFM to Shifted CEV:** We obtain a unidimensional diffusion from the DD CEV LFM;
2. **Analytical Pricing:** We obtain an analytical formula for swaptions pricing;
3. **Volatility Approximation:** In this section we obtain an approximation for the volatility term to be used in the closed form pricing formula;

### DD CEV LFM to shifted CEV

#### Random Time Change

We recall the final diffusion approximation:

We introduce a random time change defined as:

Drawing from Oksendal(2000)[[2]](#footnote-36), we can represent our multidimensional diffusion as:

where is a one dimensional Brownian motion.

The resulting one dimensional diffusion

We are therefore faced with two problems:

1. **Analytical pricing formula:** We require an analytical solution/approximation for the one dimensional shifted CEV.
2. **Volatility Approximation:** We also need to find the value/ approximation of the term.

The coming sections will address each of these problems.

### Analytical Pricing Formula

A relationship between the CEV process and the non-central chi-square distribution was first established by Schroder(1998.)

We use a simple application of the Ito’s lemma to arrive at the same result for the displaced diffusion CEV.

We begin by recalling the shifted CEV process:

We can perform a change of variables and define the variable such that:

Applying Ito’s lemma, we can define as a square root process:

This equation is a squared Bessel process process with degress of freedom.

#### The Feller Classification

Feller(1951) studies diffusions of the class:

whose corresponding Fokker-Planck equation is:

where is the Dirac delta function.

Clearly, setting , and yields our BESQ equation.

#### Solution

The quantity is the probability that conditional on

After lengthy calculation elaborated in Brecher & Lindsay (2010), we obtain the value of a swaption to be:

where:

and:

We obtain the final result to be:

### Volatility Approximation

From the formulas provided above, the most important missing parameter is the . Under the LSM, this parameter would have been readily available as the norm of the terms.

However, seeing that the LFM and LSM are incompatible, we are required to use an approximation.

In this section, we give a brief overview of the derivation of this approximation. For full calculation cf. [Volatility Dynamics & Rebonato Approximation].

We begin with a definition by Andersen and Andreasen (1997) of volatility term as:

for the vector of *“frozen”* scalar weights and the matrix of all vector functions .

We can expand this expression to:

At this juncture it is important to note that is of the same dimension as the Brownian motion i.e.  is d-dimensional.

It is also important to note that are the frozen weights .

We therefore expand the inner product to obtain:

At this point, we can freeze the terms for the final expression:

This expression is the **Rebonato approximation of the swaption LFM volatility term.**

The **Rebonato approximation** is one of the most commonly used in the market:

# Calibration

## Objective Function

We begin by recalling the volatility term

We note the large number of parameters required in the calibration i.e. (all the terms along with their corresponding correlation structure.)

We can greatly reduce the number of parameters by introducing a parametrization of the volatility surface.

We introduce a sequence where:

In this case, we have 2 brownian motions allowing us to express the terms:

At this point, it is important to note that instead of directly calibrating the shift term , we will select a number of values and test the stability of our parameters.

From the above parametrization, we can define the set of our parameters , and our optimization problem:

for a loss function .

## Optimization Algorithm (Levenberg Marquardt Algorithm)

In this section, we detail the Levenberg Marquardt Algorithm allowing us to minimize the following function:

In our case specifically, represents the parameter set and is a vector function whose result is the residual errors between the market price and calculated value from the model.

Explicitly, our functions are defined as:

The objective of the algorithm being to ensure that for a well chosen sequence , .

The basis of the component functions , therefore is:

where is the jacobian matrix at point x which can be approximated in the following manner:

for sufficiently small and a unit vector in the direction of j.

We note particularly that the sequence is constructed at each step (starting from an appropriate starting value ) such that:

The objective is to minimize the function:

The is the control term, penalising timesteps that are too big.

This allows us to find the solution that solves the equation:

### Points to note

1. The algorithm interpolates between the Gauss Newton and Gradient Descent methods. The tuning parameter is large where the objective function requires rapid minimization in large steps. As the function approaches its minimum, the parameter decreases to give way to a quasi-Gauss Newton algorithm.
2. It is important not only to fix tolerance parameters but also exit procedures when the algorithm fails to converge.

# Annex

## Andersen & Andreasen Approximation

Despite the initial method having been presented by Andersen & Andreasen, a more elaborate proof was provided by Hull & White.

We recall the definition of the swap rate in terms of the forward rates:

Using this definition, we can define the diffusion of the swap in the following manner:

where the drift is defined by the specific measure chosen. This is obtained by applying Ito’s lemma on

We apply Ito’s lemma to to obtain:

We then compute:

To show this, we consider 2 cases:

**1) Let k < j**

We denote this final value .

**2) Let k**  **j**

Test

1. For full demonstration cf. [Andersen & Andreasen Approximation](#andersen-andreasen-approximation) [↑](#footnote-ref-31)
2. Section 8.5 - Random Time Change [↑](#footnote-ref-36)